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NON-DISCRETE COMPLEX HYPERBOLIC TRIANGLE GROUPS OF TYPE (m, m, ∞)

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Abstract. In this note we prove that a complex hyperbolic triangle group of type (m, m, ∞), i.e. a group of isometries of the complex hyperbolic plane, generated by complex reflections in three complex geodesics meeting at angles π/m, π/m and 0, is not discrete if the product of the three generators is regular elliptic.

1. Introduction

We study representations of real hyperbolic triangle groups, i.e. groups generated by reflections in the sides of triangles in $H^2_{\mathbb{R}}$, in the holomorphic isometry group $\mathrm{PU}(2,1)$ of the complex hyperbolic plane $H^2_{\mathbb{C}}$.

For the basic notions of complex hyperbolic geometry, especially for the complex hyperbolic plane $H^2_{\mathbb{C}}$, see for example section 2 in [Pra05]. The general references on complex hyperbolic geometry are [Gol99, Par03].

We use the following terminology: A *complex hyperbolic triangle* is a triple (C_1, C_2, C_3) of complex geodesics in $H^2_{\mathbb{C}}$. If the complex geodesics C_{k-1} and C_{k+1} meet at the angle π/p_k we call the triangle (C_1, C_2, C_3) a (p_1, p_2, p_3)-*triangle*.

We call a subgroup of $\mathrm{PU}(2,1)$ generated by complex reflections ι_k in the sides C_k of a complex hyperbolic (p_1, p_2, p_3)-triangle (C_1, C_2, C_3) a (p_1, p_2, p_3)-*triangle group*. A (p_1, p_2, p_3)-*representation* is a representation of the group

$$\Gamma(p_1, p_2, p_3) = \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_k^2 = (\gamma_{k-1}\gamma_{k+1})^{p_k} = 1 \text{ for all } k \in \{1, 2, 3\} \rangle,$$

where $\gamma_{k+3} = \gamma_k$, and the relation $(\gamma_{k-1}\gamma_{k+1})^{p_k} = 1$ is to omit for $p_k = \infty$, into the group $\mathrm{PU}(2,1)$, given by taking the generators γ_k of $\Gamma(p_1, p_2, p_3)$ to the generators ι_k of a (p_1, p_2, p_3)-triangle group.

We prove in this paper the following result:

Theorem. *An (m, m, ∞)-triangle group is not discrete if the product of the three generators is regular elliptic.*

More about the recent developments in the study of non-discrete complex hyperbolic triangle groups of type (m, m, ∞) can be found in [Kam07], [KPT09].

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2. Non-Discreteness Proof

For fixed (p_1, p_2, p_3) the space of complex hyperbolic triangle groups is of real dimension one. We now describe a parameterisation of the space of complex hyperbolic triangles in $H^2_{\mathbb{C}}$ by means of an angular invariant α. See section 3 in [Pra05] for details.

Let c_k be the normalised polar vector of the complex geodesic C_k. Let $r_k = |\langle c_{k-1}, c_{k+1}\rangle|$. If the complex geodesics C_{k-1} and C_{k+1} meet at the angle φ_k, then $r_k = \cos\varphi_k$. We define the *angular invariant* α of the triangle (C_1, C_2, C_3) by

$$\alpha = \arg\left(\prod_{k=1}^{3}\langle c_{k-1}, c_{k+1}\rangle\right).$$

A complex hyperbolic triangle in $H^2_{\mathbb{C}}$ is determined uniquely up to isometry by the three angles and the angular invariant α (compare proposition 1 in [Pra05]). Let $\iota_k = \iota_{C_k}$ be the complex reflection in the complex geodesic C_k.

Let $\phi : \Gamma(p_1, p_2, p_3) \to \mathrm{PU}(2,1)$ be a complex hyperbolic triangle group representation and $G := \phi(\Gamma(p_1, p_2, p_3))$ the corresponding complex hyperbolic triangle group. Assume that γ is an element of infinite order in $\Gamma(p_1, p_2, p_3)$ and that its image $\phi(\gamma)$ in G is regular elliptic. Then there are two cases, either $\phi(\gamma)$ is of finite order, then ϕ is not injective, or $\phi(\gamma)$ is of infinite order, then ϕ is not discrete because the subgroup of G generated by $\phi(\gamma)$ is not discrete.

We shall show, that if the element $\iota_1\iota_2\iota_3$ is regular elliptic, then it is not of finite order, hence the corresponding triangle group is not discrete.

This statement was proved in [Sch01] for ideal triangle groups, i.e. groups of type (∞, ∞, ∞). The statement for (m, m, ∞)-triangle groups was formulated in [WG00] (Lemma 3.4.0.19), but the proof there had a gap.

Theorem. *An (m, m, ∞)-triangle group is not discrete if the product of the three generators is regular elliptic.*

Proof. We assume that the element $\iota_1\iota_2\iota_3$ is regular elliptic of finite order. Let $\tau \neq -1$ be the trace of the corresponding matrix in $\mathrm{SU}(2,1)$. The eigenvalues of this matrix are then three roots of unity with product equal to 1. Hence

$$\tau = \omega_n^{k_1} + \omega_n^{k_2} + \omega_n^{k_3}$$

for some k_1, k_2, and k_3 with $k_1 + k_2 + k_3 = 0$. Here $\omega_n = \exp(2\pi i/n)$ and n is taken as small as possible. On the other hand, the trace τ can be computed (see section 8 in [Pra05]) as

$$\tau = 8r_1r_2r_3e^{i\alpha} - (4(r_1^2 + r_2^2 + r_3^2) - 3).$$

Let

$$r = \cos\left(\frac{\pi}{m}\right).$$

For (m, m, ∞)-groups we have $r_1 = r_2 = r = \cos(\pi/m)$ and $r_3 = 1$, hence

$$\tau = (8r^2)e^{i\alpha} - (8r^2 + 1).$$

This equation implies that the complex number $\tau \neq -1$ lies on the circle with center in $-(8r^2 + 1)$ and radius $8r^2$, or in other words τ satisfies the equation

$$(\tau + (8r^2 + 1)) \cdot (\bar{\tau} + (8r^2 + 1)) = |\tau + (8r^2 + 1)|^2 = (8r^2)^2.$$

This implies in particular

$$\mathrm{Re}(\tau) < -1.$$

Let N be the least common multiple of n and $2m$. Let σ_k be the homomorphism of $\mathbb{Q}[\omega_N]$ given by $\sigma_k(\omega_N) = \omega_N^k$. For k relatively prime to n the restriction of σ_k to $\mathbb{Q}[\omega_n]$ is a Galois automorphism.

Lemma 1. *Let $\tau = \omega_n^{k_1} + \omega_n^{k_2} + \omega_n^{k_3}$ be the trace of the matrix of $\iota_1\iota_2\iota_3$, where n is taken as small as possible. Then $\sigma_k(\tau)$ satisfies the equation*

$$|\sigma_k(\tau) + \sigma_k(8r^2) + 1| = \sigma_k(8r^2).$$

This implies in particular

$$\mathrm{Re}(\sigma_k(\tau)) \leqslant -1.$$

Proof. We have

$$\tau \in \mathbb{Q}[\omega_n] \subset \mathbb{Q}[\omega_N]$$

and

$$2r = 2\cos\left(\frac{\pi}{m}\right) = \omega_{2m} + \bar{\omega}_{2m} \in \mathbb{Q}[\omega_{2m}] \subset \mathbb{Q}[\omega_N],$$

hence the equation $|\tau+(8r^2+1)| = 8r^2$ is defined in $\mathbb{Q}[\omega_N]$. The homomorphism σ_k commutes with complex conjugation and hence maps real numbers to real numbers. Applying the homomorphism σ_k to the equation

$$(\tau + (8r^2 + 1)) \cdot (\bar{\tau} + (8r^2 + 1)) = (8r^2)^2$$

we obtain

$$(\sigma_k(\tau) + \sigma_k(8r^2 + 1))(\sigma_k(\bar{\tau}) + \sigma_k(8r^2 + 1)) = (\sigma_k(8r^2))^2.$$

This equation is equivalent to

$$|\sigma_k(\tau) + \sigma_k(8r^2) + 1|^2 = (\sigma_k(8r^2))^2.$$

Since $\sigma_k(2r)$ is a real number, the number $\sigma_k(8r^2) = 2(\sigma_k(2r))^2$ is a non-negative real number. Hence $\sigma_k(\tau)$ satisfies the equation

$$|\sigma_k(\tau) + \sigma_k(8r^2) + 1| = \sigma_k(8r^2).$$

This equation means that the complex number $\sigma_k(\tau)$ lies on the circle with center in $-(\sigma_k(8r^2) + 1) < 0$ and radius $\sigma_k(8r^2) \geqslant 0$. This implies in particular

$$\mathrm{Re}(\sigma_k(\tau)) \leqslant -1. \quad \square$$

Lemma 2. *Let $\tau = \omega_n^{k_1} + \omega_n^{k_2} + \omega_n^{k_3}$ be the trace of the matrix of $\iota_1\iota_2\iota_3$, where n is taken as small as possible. For $i \in \{1, 2, 3\}$, let*

$$d_i = \frac{n}{\gcd(k_i, n)},$$

where gcd *is the greatest common divisor. Then*

$$\frac{1}{\varphi(d_1)} + \frac{1}{\varphi(d_2)} + \frac{1}{\varphi(d_3)} \geqslant 1.$$

Proof. According to Lemma 1,

$$\mathrm{Re}(\sigma_k(\tau)) \leqslant -1$$

for any homomorphism σ_k. Summing over all $k \in \{1, \dots, n-1\}$ relatively prime to n we obtain

$$\mathrm{Re}\left(\sum_{\substack{1\leqslant k<n\\(k,n)=1}} \sigma_k(\tau)\right) < -\varphi(n)$$

and hence

$$\left|\sum_{\substack{1\leqslant k<n\\(k,n)=1}} \sigma_k(\tau)\right| > \varphi(n).$$

Here φ is the Euler φ-function. The root of unity $\omega_n^{k_i}$ is a primitive d_i-th root of unity. The sum of all d_i-th primitive roots of unity is in $\{-1, 0, 1\}$, and hence is bounded by 1. The map $(\mathbb{Z}/n\mathbb{Z})^* \to (\mathbb{Z}/d_i\mathbb{Z})^*$ induced by the map $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/d_i\mathbb{Z}$ is surjective, and the preimage of any element in $(\mathbb{Z}/d_i\mathbb{Z})^*$ consists of $\varphi(n)/\varphi(d_i)$ elements. Hence we obtain the inequality

$$\big|\sum_{\substack{1\leqslant k<n\\(k,n)=1}} \sigma_k(\omega_n^{k_i})\big| \leqslant \frac{1}{\varphi(d_i)} \cdot \varphi(n)$$

for $i \in \{1, 2, 3\}$. From the inequalities

$$\begin{aligned}
\varphi(n) &< \big|\sum_{\substack{1\leqslant k<n\\(k,n)=1}} \sigma_k(\tau)\big| \\
&= \big|\sum_{\substack{1\leqslant k<n\\(k,n)=1}} \sigma_k(\omega_n^{k_1} + \omega_n^{k_2} + \omega_n^{k_3})\big| \\
&\leqslant \left(\frac{1}{\varphi(d_1)} + \frac{1}{\varphi(d_2)} + \frac{1}{\varphi(d_3)}\right) \cdot \varphi(n)
\end{aligned}$$

it follows

$$\frac{1}{\varphi(d_1)} + \frac{1}{\varphi(d_2)} + \frac{1}{\varphi(d_3)} > 1.$$

□

The inequality

$$\frac{1}{\varphi(d_1)} + \frac{1}{\varphi(d_2)} + \frac{1}{\varphi(d_3)} > 1$$

implies that the triple $(\varphi(d_1), \varphi(d_2), \varphi(d_3))$ is equal (up to permutation) to one of the triples

$$(1, ?, ?), \quad (2, 2, ?), \quad (2, 3, 3), \quad (2, 3, 4), \quad (2, 3, 5).$$

But for the Euler φ-function we have $\varphi(x) = 1$ for $x \in \{1, 2\}$, $\varphi(x) = 2$ for $x \in \{3, 4, 6\}$ and $\varphi(x) \geqslant 4$ for all other positive integers x.

- The triples $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$ cannot occur since $\varphi(x) \neq 3$ for any integer x.

- Let $\varphi(d_i) = 1$ for some $i \in \{1, 2, 3\}$. Without loss of generality we can assume that $\varphi(d_1) = 1$. Then $d_1 \in \{1, 2\}$, therefore $(k_1, n) \in \{n/2, n\}$ and $k_1 \equiv 0, n/2 \bmod n$. Hence $\omega_n^{k_1} \in \{1, -1\}$. If $k_1 \equiv 0$ then $k_2 + k_3 \equiv 0$. Let $k = k_2$, then $k_3 \equiv -k$ and
$$\tau = \omega_n^{k_1} + \omega_n^{k_2} + \omega_n^{k_3} = 1 + \omega_n^{k} + \omega_n^{-k} = 1 + 2\cos(2\pi k/n)$$
and $\mathrm{Re}(\tau) = 1 + 2\cos(2\pi k/n) \geqslant -1$ in contradiction to $\mathrm{Re}(\tau) < -1$. If $k_1 \equiv n/2$ then $k_2 + k_3 \equiv -n/2$. Let $k = k_2$, then $k_3 \equiv -k - n/2$ and
$$\tau = \omega_n^{k_1} + \omega_n^{k_2} + \omega_n^{k_3} = -1 + \omega_n^{k} - \omega_n^{-k} = -1 + 2i\sin(2\pi k/n)$$
and $\mathrm{Re}(\tau) = -1$ in contradiction to $\mathrm{Re}(\tau) < -1$.
- If $\varphi(d_i) = \varphi(d_j) = 2$ for $i, j \in \{1, 2, 3\}$, $i \neq j$, then $d_i, d_j \in \{3, 4, 6\}$, therefore $(k_i, n) \in \{n/6, n/4, n/3\}$ and $k_i \equiv \pm n/6, \pm n/4, \pm n/3 \bmod n$. Hence $\omega_n^{k_i} \in \{\alpha^{\pm 2}, \alpha^{\pm 3}, \alpha^{\pm 4}\}$, where $\alpha = \omega_{12} = \exp(2\pi i/12)$, and
$$\tau = \alpha^p + \alpha^q + \alpha^r, \quad p + q + r = 0, \quad p, q \in \{\pm 2, \pm 3, \pm 4\}.$$
Using $\mathrm{Re}(\tau) < -1$ and $\mathrm{Re}(\alpha^r) \geqslant -1$ we obtain
$$\mathrm{Re}(\alpha^p + \alpha^q) = \mathrm{Re}(\tau) - \mathrm{Re}(\alpha^r) < -1 + 1 = 0.$$
Since $\mathrm{Re}(\alpha^{\pm 2}) = \frac{1}{2}$, $\mathrm{Re}(\alpha^{\pm 3}) = 0$ and $\mathrm{Re}(\alpha^{\pm 4}) = -\frac{1}{2}$, we can only have $\mathrm{Re}(\alpha^p + \alpha^q) < 0$ if $\alpha^p + \alpha^q = \alpha^{\pm 3} + \alpha^{\pm 4}$ or $\alpha^p + \alpha^q = \alpha^{\pm 4} + \alpha^{\pm 4}$. Out of these cases, we easily check that $\mathrm{Re}(\alpha^p + \alpha^q + \alpha^{-p-q}) < -1$ holds only if $\alpha^p = \alpha^q = \alpha^4$ or $\alpha^p = \alpha^q = \alpha^{-4}$, i.e. if $\tau = 3\alpha^{\pm 4}$, but then a suitable homomorphism σ_k has the property $\mathrm{Re}(\sigma_k(\tau)) > -1$ in contradiction to Lemma 1.

□

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